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Finite-range spin glasses in the Kac limit: free energy and local observables

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Abstract

We study a finite-range spin-glass model in arbitrary dimension, where the intensity of the coupling between spins decays to zero over some distance γ^{-1}. We prove that, under a positivity condition for the interaction potential, the infinite-volume free energy of the system converges to that of the Sherrington–Kirkpatrick model, in the Kac limit $\gamma \to 0$. We study the implication of this convergence for the local order parameter, i.e., the local overlap distribution function and a family of susceptibilities associated with it, and we show that locally the system behaves like its mean field analogue. Similar results are obtained for models with p-spin interactions. Finally, we discuss a possible approach to the problem of the existence of long-range order for finite γ, based on a large deviation functional for overlap profiles. This will be developed in future work.

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1. Introduction

Since their first introduction by van der Waals in the second half of the 19th century [1], mean field theories have offered a simplified setting to understand the complex collective phenomena underlying phase transitions and low-temperature ordering. These theories, it appeared very soon, are plagued by several pathologies, which can be traced back to the fact that the finite-range character of the interactions is neglected. For example, mean field theory predicts the possibility of low-temperature ordering, independently of the space dimensionality. In addition, the free energy, in principle convex in an extended system, can present non-convexities at mean field level. In a series of classical papers Kac *et al* [2] stressed the

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role of the interaction range in these pathologies. Using a one-dimensional model for liquid–vapour transition they showed that in the so-called Kac limit [3], when the range of interaction γ^{-1} is set to infinity *after* the thermodynamic limit, while the total interaction strength is kept constant, one recovers the van der Waals theory complemented by the Maxwell construction, thus eliminating the unphysical non-convexity of the thermodynamic potential. It was also shown that coherently with general principles, while the free energy for finite γ is close to the corresponding mean field value, the phase transition only appears for $\gamma^{-1} \to \infty$. The comprehension of the question was considerably extended by Lebowitz and Penrose [4], who could study the Kac limit in great generality for any value of the spatial dimension, confirming the validity of mean field theory with the Maxwell construction. The last decade has seen a renewed interest in Kac models, which have been used as a starting point for rigorous expansions around mean field. Remarkable progress has been achieved in the analysis of models with large but finite range of interaction, both for systems without quenched disorder (ferromagnets [5, 6] and liquid models [7]) and with quenched disorder such as the random field Ising model [8] and the Hopfield model [9].

A case where the application of the mean field theory to spatially extended systems is particularly controversial is that of spin glasses. In that case, the mean field theory based on the Parisi solution [10] of the long-range interaction model of Sherrington and Kirkpatrick (SK) [11], predicts a low-temperature glassy phase with ergodicity breaking, not associated with any physical symmetry breaking. In particular, there is a transition to a low-temperature non-ergodic phase even in the presence of a field breaking explicitly the spin reversal symmetry of the Hamiltonian. This results in an organization of the low free energy states which gives rise to complex statistical properties of the so-called overlap distribution function, describing the probability law, induced by the Gibbs measure and the quenched disordered couplings, of the scalar product among configurations. This picture, correct for the mean field SK model, has been challenged, in the case of finite-dimensional models, by phenomenological theories based on scaling arguments—the droplet picture [12, 13] of the spin-glass phase—that predict the absence of the Parisi ordering in any finite dimension. According to this point of view, the spin-glass transition just corresponds to spin-reversal symmetry breaking and consequently cannot be present in a magnetic field. Years of theoretical debate, experimental results, accurate numerical simulations and even rigorous arguments [14] have failed to give a conclusive answer to the question.

It is natural in this context to look at spin glasses with Kac-type interactions, as a tool to study the relation between mean field and finite range. Spin glasses with Kac interactions were to our knowledge defined in [15], where it was shown that their free energy converges in the Kac limit to the mean field value for sufficiently high temperature, and later considered in [16], where the result was extended up to the critical temperature $T_c = 1$ of the SK model. Little progress was made until the introduction of interpolating techniques by Guerra and Toninelli in the rigorous study of disordered systems. Through those methods, it was first proved that the Kac free energy is, for a large class of Kac potentials, bounded from below by the one of the SK model [17]. Finally, joining the interpolating technique with the idea of dividing the system into large blocks where it essentially behaves like the mean field one [4], it was possible to show the convergence of the free energy to the SK one for all temperatures [18].

The scope of this paper is to give full details of the proofs of the results in [17, 18] and to discuss the implication of the Kac limit for local quantities. In particular, we will discuss the properties of the local overlap probability distribution and local susceptibilities and show that for small γ they are close, respectively, to the overlap probability distribution and susceptibilities of the SK model at the same temperature. As a further extension, we generalize these results to the case of Kac spin-glass models with p-body interactions, where p is an even integer.

The present work is organized as follows: in section 2 we define the models and state the main results. In section 3, we prove convergence of the free energy to the mean field limit, when $\gamma \to 0$, while in section 4 we prove convergence for the distribution of the local overlaps. Finally, in section 5 we outline a possible strategy to study the occurrence of long-range order in finite-range spin glasses, when γ is small but finite. This requires the introduction of a large deviation functional for the profile of two replicas. This strategy will be pursued in a forthcoming paper [19].

2. The models and the main results

2.1. Spin glasses with Kac-type interactions

In this section, we define a finite-range version of the p-spin spin-glass model well suited to study the Kac limit. We recall that the original infinite-range model is defined [20], for an integer p and some magnetic field $h \in \mathbb{R}$, by a set of N Ising spins $\sigma_i = \pm 1, i = 1, \ldots, N$ interacting via the Hamiltonian:

$$H_N^{(p)}(\sigma, h; J) = -\sqrt{\frac{p!}{2N^{p-1}}} \sum_{1 \leqslant i_1 < \cdots < i_p \leqslant N} J_{i_1,\ldots,i_p} \sigma_{i_1} \cdots \sigma_{i_p} - h \sum_{i=1}^{N} \sigma_i \qquad (1)$$

where the couplings $J_{i_1,\ldots,i_p}$ are independent identically distributed (i.i.d.) Gaussian random variables with averages

$$E J_{i_1,\ldots,i_p} = 0 \qquad E J^2_{i_1,\ldots,i_p} = 1. \qquad (2)$$

For $p = 2$, one recovers the usual SK model.

We propose a generalization of the model to the d-dimensional lattice $\mathbb{Z}^d$. Throughout all the paper, we will assume periodic boundary conditions for convenience, in order to ensure translation invariance for the quenched averages. Therefore, we will always consider the system on T_L, the d-dimensional discrete torus of side L and cardinality $N = L^d$. Given $\gamma > 0$ and a family $J_{i_1,\ldots,i_p}, i_r \in T_L, r = 1, \ldots, p$ of i.i.d. Gaussian random variables with averages as in (2), we define the finite-volume Hamiltonian for the p-spin spin glass with Kac-type interactions as

$$\begin{aligned} H_L^{(p,\gamma)}(\sigma, h; J) &= -K_L^{(p,\gamma)}(\sigma; J) - h \sum_{i \in T_L} \sigma_i \\ &= -\sum_{i_1,\ldots,i_p \in T_L} \sqrt{w^{(p)}(i_1, \ldots, i_p; \gamma)} J_{i_1,\ldots,i_p} \sigma_{i_1} \cdots \sigma_{i_p} - h \sum_{i \in T_L} \sigma_i. \end{aligned} \qquad (3)$$

Here,

$$w^{(p)}(i_1, \ldots, i_p; \gamma) = \frac{\sum_{k \in T_L} \psi(\gamma |i_1 - k|) \cdots \psi(\gamma |i_p - k|)}{W(\gamma)^{p/2}} \qquad (4)$$

and

$$W(\gamma) = \left(\sum_{k \in T_L} \psi(\gamma |k|) \right)^2 \qquad (5)$$

where $\psi(|x|), x \in \mathbb{R}^d$, is a non-negative function with compact support,

$$\psi(|x|) = 0 \qquad \text{if} \quad |x| \geqslant 1$$

sufficiently regular to be Riemann integrable.

For $\gamma \simeq 0$, it can immediately be seen that one recovers the usual form [4] for the Kac potential, i.e.,

$$w^{(2)}(i,j;\gamma) \simeq \gamma^d \phi(\gamma|i-j|) \tag{6}$$

where in our case

$$\phi(|i-j|) = \frac{\int \psi(|i-k|)\psi(|j-k|)\,\mathrm{d}^d k}{\left(\int \psi(|k|)\,\mathrm{d}^d k\right)^2}. \tag{7}$$

On the other hand, the reason for the particular choice (4) is that it guarantees that $w^{(2)}$ is positive definite, i.e.,

$$\sum_{i,j\in T_L} w^{(2)}(i,j;\gamma) v_i v_j \geqslant 0 \qquad \forall\{v_i\}_{i\in T_L} \quad v_i \in \mathbb{R} \tag{8}$$

and that a suitable positive definiteness property, implied by lemma 2 below, is satisfied by $w^{(p)}$, for any even p. This property will turn out to be essential in proving our results.

It is easy to realize that the potentials $w^{(p)}$ satisfy the following properties:

1. invariance with respect to translations on the torus:

$$w^{(p)}(i_1,\ldots,i_p;\gamma) = w^{(p)}(i_1+k,\ldots,i_p+k;\gamma) \qquad \forall k \in T_L \tag{9}$$

2. finite range of order $\xi = 1/\gamma$:

$$w^{(p)}(i_1,\ldots,i_p;\gamma) = 0 \qquad \text{if} \quad \exists\, a,b : |i_a - i_b| \geqslant 2\xi \tag{10}$$

3. normalization:

$$\sum_{i_2,\ldots,i_p\in T_L} w^{(p)}(i_1,\ldots,i_p;\gamma) = 1 \qquad \forall i_1 \in T_L \tag{11}$$

and

$$\sum_{i_{r+1},\ldots,i_p\in T_L} w^{(p)}(i_1,\ldots,i_p;\gamma) = w^{(r)}(i_1,\ldots,i_r;\gamma)$$
$$\forall i_1,\ldots,i_r \in T_L \quad \text{for} \quad 1 < r < p \tag{12}$$

besides of course symmetry with respect to index permutation. Note that $w^{(1)}(i;\gamma) = 1$. Note also that, while $w^{(p)}(i_1,\ldots,i_p;\gamma)$ in principle depends on the size of the system, this dependence disappears as soon as $L > 4/\gamma$ and is therefore inessential in view of the fact that we consider the limit $\gamma \to 0$ only after $L \to \infty$.

Remark. All the results of the present paper extend to the case where $\psi(|x|)$ is only assumed to decay sufficiently fast for $|x| \to \infty$ so that

$$\psi(|x|) \leqslant C|x|^{-d-\delta} \tag{13}$$

for some $\delta, C > 0$. A similar condition was required in [4].

For a given inverse temperature β, we denote as $Z_L^{(p,\gamma)}(\beta,h;J)$ the disorder-dependent partition function of the model (1), by $\langle\cdot\rangle$ the corresponding Gibbs average, and by $f_L^{(p,\gamma)}(\beta,h)$ the finite-volume quenched free energy

$$f_L^{(p,\gamma)}(\beta,h) = -\frac{1}{\beta L^d} E \ln Z_L^{(p,\gamma)}(\beta,h;J). \tag{14}$$

Our first result shows that, as a generalization of [17] and [18], the free energy of the Kac model tends, for $\gamma \to 0$, to that of the corresponding mean field model. Let us first of all redefine the Hamiltonian (1) in a slightly different way, in order to make the comparison with model (1) more straightforward:

$$H_L^{(p)}(\sigma, h; J) = -K_L^{(p)}(\sigma; J) - h\sum_{i\in T_L}\sigma_i = -\sum_{i_1,\ldots,i_p\in T_L}\frac{J_{i_1,\ldots,i_p}}{L^{d(p-1)/2}}\sigma_{i_1}\cdots\sigma_{i_p} - h\sum_{i\in T_L}\sigma_i \quad (15)$$

whose partition function will be denoted as $Z_L^{(p)}(\beta, h; J)$. Of course, in this case the geometrical structure of the lattice and the space dimensionality are completely inessential. Then, the following holds.

Theorem 1. *For any β, h and p even, and for any choice of the function ψ in (4), the following limit exists and satisfies:*

$$\lim_{\gamma\to 0} f^{(p,\gamma)}(\beta, h) \equiv \lim_{\gamma\to 0}\lim_{L\to\infty} f_L^{(p,\gamma)}(\beta, h) = f^{(p)}(\beta, h) \quad (16)$$

where $f^{(p)}(\beta, h)$ is the infinite-volume quenched free energy for the mean field p-spin spin glass with Hamiltonian (15), i.e.,

$$f^{(p)}(\beta, h) = \lim_{L\to\infty} f_L^{(p)}(\beta, h) \equiv -\lim_{L\to\infty}\frac{1}{\beta L^d} E \ln Z_L^{(p)}(\beta, h; J). \quad (17)$$

Remark. The existence of the infinite-volume free energy $f^{(p,\gamma)}(\beta, h)$ for the model (3) can be proved along the lines of [21], while for the mean field model (15) the analogous result was proved in [22]. It is interesting to remark that the methods we develop in the present paper can be employed to obtain a new proof of the existence of the limit (17), see end of section 3.

2.2. Local observables

The next natural question after proving the convergence of the free energy to its mean field limit, is to investigate in which sense the system has a behaviour close to mean field for small but finite γ. In particular, we would like to have information on the overlap distribution function and its associated susceptibilities, whose non-trivial behaviour characterizes the Parisi order in mean field theory. As a first step, we investigate the behaviour of 'local' observables on the scale $1/\gamma$, for γ small. We will show that on this scale the behaviour of the system is close to that of the mean field model, in any dimension. The natural approach is to add suitable perturbations to the Hamiltonian, such that the averages of the observables of interest are given by the derivatives of the free energy with respect to the perturbing parameters.

Given two configurations σ^1, σ^2 of the system and $k \in T_L$, let us define the 'local overlap'

$$q_k^\gamma(\sigma^1, \sigma^2) = \sum_{i\in T_L}\frac{\psi(\gamma|i-k|)}{W(\gamma)^{1/2}}\sigma_i^1\sigma_i^2. \quad (18)$$

In particular, we will have in mind the case where ψ is the Heaviside function

$$\psi(|x|) = 1_{\{|x|_\infty\leqslant 1\}} \quad (19)$$

$|x|_\infty$ being defined as $\max_{r=1,\ldots,d}|x_r|$. In this case, $W(\gamma) \simeq (2/\gamma)^{2d}$ and equation (18) reduces to the familiar definition

$$q_k^\gamma(\sigma^1, \sigma^2) = \left(\frac{\gamma}{2}\right)^d \sum_{i\in T_L:|i-k|_\infty\leqslant 1/\gamma}\sigma_i^1\sigma_i^2 \quad (20)$$

of the overlap in the cube of side $2\gamma^{-1}$ centred at the site k. The Gibbs measure and the quenched couplings induce the probability distribution $P_L^{(p,\gamma)}(q)$ for the overlap, depending also on β and h, which we write formally as

$$P_L^{(p,\gamma)}(q) = E\left(\frac{\sum_{\sigma^1,\sigma^2} \exp\left(-\beta H_L^{(p,\gamma)}(\sigma^1, h; J) - \beta H_L^{(p,\gamma)}(\sigma^2, h; J)\right)\delta\left(q_k^\gamma(\sigma^1,\sigma^2) - q\right)}{\left(Z_L^{(p,\gamma)}(\beta, h; J)\right)^2}\right). \tag{21}$$

As in [23], we consider a family of perturbations associated with the function (21), and introduce the Hamiltonian

$$H_L^{(p,\gamma)}(\sigma, \{\varepsilon\}, h; J) = -K_L^{(p,\gamma)}(\sigma; J) - \sum_{r\geqslant 1} \varepsilon_r K_L^{(r,\gamma)}(\sigma; J^{(r)}) - h\sum_{i\in T_L} \sigma_i \tag{22}$$

where the families of Gaussian random variables $J^{(r)}$ are independent for different r, and the real numbers ε_r decay to zero sufficiently rapidly when $r \to \infty$, to ensure that the corresponding free energy is finite. Actually, we will require that $\{\varepsilon_r\}_{r\geqslant 1}$ belongs to a region $R_p \subset \mathbb{R}^\infty$, defined precisely in appendix B, characterised by the fact that not only $|\varepsilon_r|$ are sufficiently small for r large, but also that

$$|\varepsilon_s| \ll |\varepsilon_2| \qquad s \geqslant 3. \tag{23}$$

In any case, we will be eventually interested in letting *all* the ε_r approach zero, so that the corresponding terms in the Hamiltonian are actually small perturbations.

Let $P_{L,\varepsilon}^{(p,\gamma)}(q)$ be the probability distribution density of $q_{12}^\gamma \equiv q_0^\gamma(\sigma^1,\sigma^2)$ in the presence of the perturbations. The role of the perturbations in (22) is easily understood. Indeed, we will see that

$$\partial_{\varepsilon_r} f_L^{(p,\gamma)}(\beta, \{\varepsilon\}, h) = -\beta\varepsilon_r\left(1 - E\left\langle\left(q_{12}^\gamma\right)^r\right\rangle\right) = -\beta\varepsilon_r\left(1 - \int \mathrm{d}q\, P_{L,\varepsilon}^{(p,\gamma)}(q) q^r\right) \tag{24}$$

where $f_L^{(p,\gamma)}(\beta, \{\varepsilon\}, h)$ is the finite-volume free energy of the model (22). Then, one has the following theorem.

Theorem 2. *For any choice of ψ in (4), and for almost every value of $\{\varepsilon\} \in R_p$, the following limit exists and*

$$\lim_{\gamma\to 0}\lim_{L\to\infty} P_{L,\varepsilon}^{(p,\gamma)}(q) = P_\varepsilon^{(p)}(q) = \lim_{L\to\infty} P_\varepsilon^{(p)}(q) \tag{25}$$

where the rhs is the infinite-volume quenched average of the probability distribution of the full overlap

$$q_{12} = L^{-d}\sum_{i\in T_L} \sigma_i^1\sigma_i^2 \tag{26}$$

for the 'perturbed p-spin mean field model' defined by the Hamiltonian

$$H_L^{(p)}(\sigma, \{\varepsilon\}, h; J) = -K_L^{(p)}(\sigma; J) - \sum_{r\geqslant 1} \varepsilon_r K_L^{(r)}(\sigma; J^{(r)}) - h\sum_{i\in T_L}\sigma_i. \tag{27}$$

The convergence in (25) is in the sense that, for any continuous function $f(\cdot)$,

$$\lim_{\gamma\to 0}\lim_{L\to\infty}\int f(q) P_{L,\varepsilon}^{(p,\gamma)}(q)\,\mathrm{d}q = \lim_{L\to\infty}\int f(q) P_{L,\varepsilon}^{(p)}(q)\,\mathrm{d}q.$$

Remarks. Theorem 2 shows that for small but finite γ, on the scale of the range of the interaction, the overlap probability distribution is close to the corresponding mean field one at the same temperature, which is known to be non-trivial at low temperature. Of course, as already happens in non-disordered models, this does not have implications on the possible presence of long-range order.

As is well known [10], at the mean field level the probability distribution of a single overlap is not sufficient to give a full description of a spin-glass model, and the knowledge of the joint distribution $P(\{q_{ab}\})$ of the overlaps q_{ab} among any two replicas a, b is required. Parisi's theory for the mean field model predicts a highly non-trivial *ultrametric* structure for $P(\{q_{ab}\})$, whose validity has not been rigorously established yet. It would be possible to extend the ideas of the present section, and to show that the joint distribution of the local overlaps between several replicas behaves, for $\gamma \to 0$, like the corresponding one for the mean field model, but we will not pursue this. In this case, the perturbations to be added are the Kac-type analogues of those introduced in [24].

Finally, let us remark that the convergence in (25) is proved to hold only for *almost every* $\{\varepsilon\}$, so that nothing can be said *a priori* for $\varepsilon \equiv 0$, corresponding to the original model (3), since the Gibbs averages need not in principle be *stochastically stable*, i.e., continuous with respect to the perturbing parameters. Despite the fact that the point $\varepsilon = 0$ does not appear to have anything special (apart from corresponding to a spin reversal symmetric Hamiltonian), general techniques to prove stochastic stability are at present unknown. The same problem arises also at the level of mean field theory (see for instance [25]), where stochastic stability has been related to continuity of the Gibbs state under temperature changes [26] and is a drawback of the method of stochastic perturbations.

3. Kac limit for the free energy

3.1. Proof of theorem 1

First of all, extending the ideas of [17], we will prove the lower bound

$$f_L^{(p,\gamma)}(\beta, h) \geqslant f_L^{(p)}(\beta, h) \tag{28}$$

uniformly in $\gamma > 0$ and $L > 4/\gamma$. To this purpose, let

$$Z_L(t) = \exp\, \beta \left(\sqrt{t} K_L^{(p,\gamma)}(\sigma; J) + \sqrt{1-t}\, K_L^{(p)}(\sigma; J') + h \sum_{i \in T_L} \sigma_i \right) \tag{29}$$

where the families of Gaussian variables J and J' are mutually independent. Then,

$$-\frac{1}{\beta L^d} E \ln Z_L(0) = f_L^{(p)}(\beta, h) \tag{30}$$

and

$$-\frac{1}{\beta L^d} E \ln Z_L(1) = f_L^{(p,\gamma)}(\beta, h). \tag{31}$$

The t derivative of the free energy can be written, introducing two replicas σ^1, σ^2 of the system with the same disorder realization, as

$$-\frac{\mathrm{d}}{\mathrm{d}t} \frac{1}{\beta L^d} E \ln Z_L(t) = \frac{\beta}{2} E \left(\sum_{i_1,\ldots,i_p \in T_L} \frac{w^{(p)}(i_1, \ldots, i_p; \gamma)}{L^d} \left\langle \sigma_{i_1}^1 \sigma_{i_1}^2 \ldots \sigma_{i_p}^1 \sigma_{i_p}^2 \right\rangle - \left\langle q_{12}^p \right\rangle \right) \tag{32}$$

where q_{12} is the overlap defined in (26) and the t dependence is implicit in the Gibbs average $\langle \cdot \rangle$. In (32) we employed property (11), which holds as soon as $L > 4/\gamma$. Thanks to lemma 2 below and to the translation invariance of the quenched averages, ensured by the periodic boundary conditions, (32) equals

$$\frac{\beta}{2}\sum_{r=2}^{p}\binom{p}{r}E\left\langle q_{12}^{p-r}\left(\sum_{i\in T_L}\frac{\psi(\gamma|i|)}{W(\gamma)^{1/2}}\left(\sigma_i^1\sigma_i^2-q_{12}\right)\right)^r\right\rangle. \tag{33}$$

Now, it can immediately be seen that

$$\sum_{r=2}^{p}\binom{p}{r}x^{p-r}y^r=(x+y)^p-x^p-px^{p-1}y\geqslant 0 \tag{34}$$

for any $x, y \in \mathbb{R}$ and p even, so that the derivative in (33) is non-negative, and (28) follows.

On the other hand, the idea to obtain the upper bound

$$\limsup_{\gamma\to 0} f^{(p,\gamma)}(\beta,h)\leqslant f^{(p)}(\beta,h) \tag{35}$$

is to interpolate between the Kac model in the box T_L and a system made of a collection of many independent mean field subsystems enclosed in cubes of side ℓ [18] . The crucial point, as in [4], is to choose

$$1\ll \ell \ll 1/\gamma \ll L \tag{36}$$

and to let the three lengths diverge in this order. Let us divide T_L into sub-cubes Ω_n of side $\ell, n = 1, \ldots, (L/\ell)^d$, and introduce the partition function

$$Z_L(t)=\sum_\sigma \exp\beta\left(\sqrt{t}K_L^{(p,\gamma)}(\sigma;J)+\sqrt{1-t}\sum_n\sum_{i_1,\ldots,i_p\in\Omega_n}\frac{J'_{i_1,\ldots,i_p}}{\ell^{d(p-1)/2}}\sigma_{i_1}\cdots\sigma_{i_p}+h\sum_{i\in T_L}\sigma_i\right) \tag{37}$$

which interpolates between the Kac model and a collection of many non-interacting mean field models in the different boxes, with independent couplings. In this case,

$$-\frac{1}{\beta L^d}E\ \ln Z_L(0)=f_\ell^{(p)}(\beta,h) \tag{38}$$

and

$$-\frac{1}{\beta L^d}E\ \ln Z_L(1)=f_L^{(p,\gamma)}(\beta,h) \tag{39}$$

while the t derivative of the free energy is

$$-\frac{\mathrm{d}}{\mathrm{d}t}\frac{1}{\beta L^d}E\ \ln Z_L(t)$$
$$=\frac{\beta}{2}E\left(\sum_{i_1,\ldots,i_p\in T_L}\frac{w^{(p)}(i_1,\ldots,i_p;\gamma)}{L^d}\left\langle\sigma_{i_1}^1\sigma_{i_1}^2\cdots\sigma_{i_p}^1\sigma_{i_p}^2\right\rangle-\left(\frac{\ell}{L}\right)^d\sum_n\left\langle\left(q_{12}^{(n)}\right)^p\right\rangle\right)$$

where

$$q_{12}^{(n)}=\frac{1}{\ell^d}\sum_{i\in\Omega_n}\sigma_i^1\sigma_i^2$$

is the partial overlap referring to the nth box. Defining

$$w_+^{(p)}(n_1,\ldots,n_p;\gamma)=\max_{i_r\in\Omega_{n_r},r=1,\ldots,p}w^{(p)}(i_1,\ldots,i_p;\gamma) \tag{40}$$

one has the immediate bound

$$-\frac{\mathrm{d}}{\mathrm{d}t}\frac{1}{\beta L^d}E\ \ln\ Z_L(t) \leqslant \frac{\beta}{2}\left(\frac{\ell}{L}\right)^d \times E\left(\ell^{d(p-1)}\sum_{n_1,\dots,n_p} w_+^{(p)}(n_1,\dots,n_p;\gamma)\langle q_{12}^{(n_1)}\dots q_{12}^{(n_p)}\rangle - \sum_n \langle (q_{12}^{(n)})^p\rangle\right) \quad (41)$$

and, thanks to lemma 1 below,

$$-\frac{\mathrm{d}}{\mathrm{d}t}\frac{1}{\beta L^d}E\ \ln\ Z_L(t) \leqslant \frac{\beta}{2}\left(\frac{\ell}{L}\right)^d E\left(\ell^{d(p-1)}\sum_{n_1,\dots,n_p} w_+^{(p)}(n_1,\dots,n_p;\gamma)\langle (q_{12}^{(n_1)})^p\rangle - \sum_n \langle (q_{12}^{(n)})^p\rangle\right). \quad (42)$$

Finally, one employs the property

$$\lim_{\gamma\to 0}\lim_{L\to\infty}\sum_{n_2,\dots,n_p} w_+^{(p)}(n_1,\dots,n_p;\gamma) = \ell^{-d(p-1)} \quad (43)$$

which holds [4] since, in the Kac limit, the potential $w^{(p)}(i_1,\dots,i_p;\gamma)$ becomes smoother and smoother, so that the sum in (43) converges to its Riemann integral, up to the factor $\ell^{d(p-1)}$ which is just the size of the cell in the Riemann sum. Therefore,

$$\limsup_{\gamma\to 0}\limsup_{L\to\infty}\frac{\mathrm{d}}{\mathrm{d}t}\frac{-1}{\beta L^d}E\ \ln\ Z_L(t) \leqslant 0 \quad (44)$$

from which (35) follows, after taking the limit $\ell \to \infty$.

Remark. The arguments outlined above can also be employed to obtain a new proof of the existence of the thermodynamic limit for the free energy of mean field spin-glass models, independent of the convexity argument developed in [22, 27, 28]. Indeed, from the first interpolation (29) above, it follows that

$$\liminf_{\gamma\to 0}\lim_{L\to\infty} f_L^{(p,\gamma)}(\beta,h) \geqslant \limsup_{L\to\infty} f_L^{(p)}(\beta,h) \quad (45)$$

while from (37) one obtains

$$\limsup_{\gamma\to 0}\lim_{L\to\infty} f_L^{(p,\gamma)}(\beta,h) \leqslant \liminf_{\ell\to\infty} f_\ell^{(p)}(\beta,h) \quad (46)$$

which together imply the existence of the limit in (17).

Lemma 1. *For any even p, and any real numbers $x_1,\dots,x_p$, one has*

$$\frac{x_1^p+\cdots+x_p^p}{p} \geqslant x_1\cdots x_p. \quad (47)$$

Proof. One has

$$\frac{x_1^p+\cdots+x_p^p}{p} \geqslant \left(\frac{|x_1|+\cdots+|x_p|}{p}\right)^p \geqslant |x_1|\cdots|x_p| \geqslant x_1\cdots x_p \quad (48)$$

where the first inequality follows from convexity of the function $x \to x^p$ and the second from the 'arithmetic–geometric' inequality [29]. □

Lemma 2. *Given numbers τ_i, $i \in T_L$, and defining*

$$M = \frac{1}{L^d} \sum_{i \in T_L} \tau_i$$

one has for $p \geqslant 1$

$$\frac{1}{L^d} \sum_{i_1,\dots,i_p \in T_L} w^{(p)}(i_1,\dots,i_p;\gamma)\tau_{i_1}\dots\tau_{i_p} - M^p$$
$$= \frac{1}{L^d} \sum_{r=2}^{p} M^{p-r} \binom{p}{r} \sum_{i_1,\dots,i_r \in T_L} w^{(r)}(i_1,\dots,i_r;\gamma) \prod_{s=1}^{r} (\tau_{i_s} - M). \quad (49)$$

The proof of lemma 2 is given in appendix A.

4. The distribution of the local overlaps

4.1. Proof of theorem 2

First of all, it is not difficult to generalize theorem 1 to the perturbed model (22), i.e., to prove that

$$\lim_{\gamma\to 0} \lim_{L\to\infty} f_L^{(p,\gamma)}(\beta, \{\varepsilon\}, h) = f^{(p)}(\beta, \{\varepsilon\}, h) \equiv \lim_{L\to\infty} f_L^{(p)}(\beta, \{\varepsilon\}, h). \quad (50)$$

The additional difficulty is the presence of the r-spin perturbations with r odd, whose effect can however be controlled, if the parameters $\{\varepsilon_r\}_{r\geqslant 1}$ belong to the region R_p mentioned in section 2.2. The proof of (50) is outlined in appendix B. Due to the convexity of the free energy, the partial derivatives with respect to the perturbing parameters exist almost everywhere and [30]

$$\lim_{\gamma\to 0} \lim_{L\to\infty} \partial_{\varepsilon_r} f_L^{(p,\gamma)}(\beta, \{\varepsilon\}, h) = \lim_{\gamma\to 0} \lim_{L\to\infty} \frac{1}{L^d} E\langle K_L^{(r,\gamma)}\rangle = \lim_{L\to\infty} \partial_{\varepsilon_r} f_L^{(p)}(\beta, \{\varepsilon\}, h) \quad (51)$$

where the thermal average $\langle\cdot\rangle$ corresponds to the Hamiltonian (22) and therefore depends also on p, γ and $\{\varepsilon\}$. Now, using translation invariance and recalling definitions (3), (18), an immediate integration by parts on the Gaussian disorder shows that

$$\frac{1}{L^d} E\langle K_L^{(r,\gamma)}\rangle = -\beta\varepsilon_r\left(1 - E\langle (q_{12}^{\gamma})^r\rangle\right). \quad (52)$$

On the other hand, the rhs of (51) gives the same expression, only with q_{12}^{γ} replaced by the full overlap q_{12}, and with the Gibbs average replaced by the Gibbs average of the mean field model (27). Therefore, one has convergence of the moments of $P_\varepsilon^{(p,\gamma)}(q)$ to those of $P_\varepsilon^{(p)}(q)$ in the Kac limit, which implies (25), since the (local) overlaps are bounded random variables.

5. From local to global order: outlook and open problems

The scope of this section is to discuss some perspectives opened by our results. The following discussion will have an informal character, with no aim to mathematical rigour.

The main point of our work is that for small γ the 'local physics' of finite-dimensional spin glasses is close to the one of mean field models. The free energy tends to that of the corresponding mean field model and, at low temperature, the Kac model exhibits locally a non-trivial distribution of the overlaps on length scales of order γ^{-1}. From the definition (18) it is clear that, given any two configurations σ^1, σ^2, the local overlap q_k^γ is a smooth function of the space index k on the scale γ^{-1}. Indeed, one has

$$\left|q_k^\gamma(\sigma_1,\sigma_2) - q_l^\gamma(\sigma_1,\sigma_2)\right| \leqslant \sum_{i\in T_L} \frac{|\psi(\gamma|i-k|) - \psi(\gamma|i-l|)|}{W(\gamma)^{1/2}} \sim \gamma|k-l|. \tag{53}$$

The Kac model is therefore an example supporting the possibility, advocated several times, that in short-range spin glasses replica symmetry breaking describes at least local-ordering properties [31–33]. The next relevant question concerns the possibility of long-range order. Of course local, short-range ordering does not necessary imply global, long-range ordering, in which the global overlaps exhibit non-trivial statistics. In low enough dimension, for all positive γ, the system should be a paramagnet at all temperatures (see [34] for the discussion of the one-dimensional case), where the overlap distribution is a single δ-function in zero. This means that the typical configurations have overlap profiles taking locally the values in the support of the mean field overlap probability distribution $P^{\mathrm{MF}}(q)$, but averaging to zero on the scale of the system size. Conversely, according to the replica symmetry breaking theory (RSB) [10], one would expect that in high enough dimension the distributions of the local and the global overlaps should coincide. This corresponds to a long-range order where the overlap profiles dominating the overlap measure are constant in space.

This observation suggests defining RSB in extended systems in terms of sensitivity to 'overlap boundary conditions'. Take two copies of the Kac spin glass with the same quenched disorder, enclosed for simplicity in the d-dimensional hypercube Λ_L of side L. Consider now a region of thickness $\sim\gamma^{-1}$ around the boundary of Λ_L and choose some value p in the support of $P^{\mathrm{MF}}(q)$. The boundary conditions will consist in constraining the spin configurations of the two systems so that their mutual local overlap q_k^γ equals p for all k in the boundary region. We say that RSB long-range order is present if the probability that the overlap $q_0(\sigma^1,\sigma^2)^\gamma$ around the central site of Λ_L is different from p vanishes in the thermodynamic limit. A mathematical theory of finite-dimensional spin glasses should find a way to estimate this probability.

Kac models could give the opportunity of studying this probability in a simplified setting, as happens for the corresponding object in the ferromagnetic case. In ferromagnetic Kac systems one can define 'block spins', i.e., local magnetization on scales l such that $1 \ll l \ll \gamma^{-1}$, and the probability of block spin profiles in space. This probability takes for small γ the form of a large deviation functional [5] with rate function consisting of the space-dependent mean field free energy as a function of the profile. This is the starting point of a 'semiclassical' analysis in which the saddle point treatment of the rate function can be used to infer the phase structure of the model for small γ, and in particular the existence of spontaneous magnetization and sensitivity to boundary conditions at low temperature.

In analogy with that, in the spin-glass case, one can introduce the probability of local overlap profiles. The formalism to study that probability within the replica method has been introduced in [35]. In that work, an unjustified saddle point procedure was used to estimate the free-energy cost of overlap spatial inhomogeneities, which suggested the presence of RSB in dimension greater than or equal to $d = 3$. Straightforward application of that formalism to the Kac spin-glass model again gives a large deviation theory where the dominant profiles are constant in space. The resulting rate function is indeed similar to the one used in [35], thus vindicating *a posteriori* the saddle point procedure used in that paper.

Of course, in order to give decisive contributions in the debate about the nature of the spin-glass phase in finite-dimensional systems, these arguments should be put on a solid mathematical basis, and the large deviation approach of [35] should receive a justification going beyond the replica theory. Progress in that direction will be reported soon [19].

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Appendix A. Proof of lemma 2

Equation (49) follows from the following more general identity: for $p \geqslant 1$ and $k = 0, \ldots, p$,

$$\frac{1}{L^d}\sum_{i_1,\ldots,i_p\in T_L} w^{(p)}(i_1,\ldots,i_p;\gamma)\tau_{i_1}\cdots\tau_{i_k}(\tau_{i_{k+1}}-M)\cdots(\tau_{i_p}-M)$$
$$= M^p\delta_{k=p} + \frac{1}{L^d}\sum_{r=\max(2,p-k)}^{p} M^{p-r}\binom{k}{k+r-p}$$
$$\times \sum_{i_1,\ldots,i_r\in T_L} w^{(r)}(i_1,\ldots,i_r;\gamma)\prod_{s=1}^{r}(\tau_{i_s}-M). \tag{A1}$$

Clearly, the identity is trivial for $k = 0$, and (49) follows taking $k = p$.

The proof of (A1) proceeds by induction on p (for $p = 1, 2$, it is trivial). Suppose the identity is true up to $p - 1$. Then, one has for $k \geqslant 1$

$$\frac{1}{L^d}\sum_{i_1,\ldots,i_p\in T_L} w^{(p)}(i_1,\ldots,i_p;\gamma)\tau_{i_1}\cdots\tau_{i_k}(\tau_{i_{k+1}}-M)\cdots(\tau_{i_p}-M)$$
$$= \frac{1}{L^d}\sum_{i_1,\ldots,i_p\in T_L} w^{(p)}(i_1,\ldots,i_p;\gamma)\tau_{i_1}\cdots\tau_{i_{k-1}}(\tau_{i_k}-M)\cdots(\tau_{i_p}-M)$$
$$+ \frac{M}{L^d}\sum_{i_1,\ldots,i_{p-1}\in T_L} w^{(p-1)}(i_1,\ldots,i_{p-1};\gamma)\tau_{i_1}\cdots\tau_{i_{k-1}}(\tau_{i_k}-M)\cdots(\tau_{i_{p-1}}-M)$$
$$= \frac{1}{L^d}\sum_{i_1,\ldots,i_p\in T_L} w^{(p)}(i_1,\ldots,i_p;\gamma)\tau_{i_1}\cdots\tau_{i_{k-1}}(\tau_{i_k}-M)\cdots(\tau_{i_p}-M)$$
$$+ M^p\delta_{k=p} + \frac{M}{L^d}\sum_{r=\max(2,p-k)}^{p-1} M^{p-r-1}\binom{k-1}{k+r-p}$$
$$\times \sum_{i_1,\ldots,i_r\in T_L} w^{(r)}(i_1,\ldots,i_r;\gamma)\prod_{s=1}^{r}(\tau_{i_s}-M) \tag{A2}$$

where in the first step we used the property (12). Repeating k times the trick of replacing τ_{i_a} by $(\tau_{i_a} - M) + M$, one can finally rewrite (A2) as

$$M^p\delta_{k=p} + \frac{1}{L^d}\sum_{i_1,\ldots,i_p\in T_L} w^{(p)}(i_1,\ldots,i_p;\gamma)\prod_{s=1}^{p}(\tau_{i_s}-M) + \sum_{w=0}^{k-1}\frac{1}{L^d}\sum_{r=\max(2,p-k+w)}^{p-1}$$
$$\times M^{p-r}\binom{k-w-1}{r-p+k-w}\sum_{i_1,\ldots,i_r\in T_L} w^{(r)}(i_1,\ldots,i_r;\gamma)\prod_{s=1}^{r}(\tau_{i_s}-M).$$

and it is not difficult to check that this coincides with the rhs of (A1), thanks to the identity

$$\sum_{w=0}^{k+r-p} \binom{k-w-1}{k+r-p-w} = \binom{k}{k+r-p}. \tag{A3}$$

Appendix B. Proof of equation (50)

We sketch the proof of equation (50), pointing out only the differences with respect to the proof in section 3. In view of obtaining the analogue of the lower bound (28), one generalizes in an obvious way the interpolation (29) and obtains, in analogy with (33), the identity

$$-\frac{\mathrm{d}}{\mathrm{d}t}\frac{1}{\beta L^d}E\ \ln Z_L(t) = \frac{\beta}{2}\sum_{r=2}^{p}\binom{p}{r}E\langle q_{12}^{p-r}y^r\rangle + \frac{\beta}{2}\varepsilon_2^2 E\langle y^2\rangle + \frac{\beta}{2}\sum_{s\geqslant 3}\varepsilon_s^2\sum_{r=2}^{s}\binom{s}{r}E\langle q_{12}^{s-r}y^r\rangle \tag{B1}$$

where

$$y = \sum_{i\in T_L}\frac{\psi(\gamma|i|)}{W(\gamma)^{1/2}}\left(\sigma_i^1\sigma_i^2 - q_{12}\right) \tag{B2}$$

so that $|y| \leqslant 2$. All the terms in (B1) vanish at least as fast as y^2, when $y \to 0$. This implies that, provided that

$$\sum_{s\geqslant 3}\varepsilon_s^2\sum_{r=2}^{s}2^r\binom{s}{r} \leqslant \varepsilon_2^2 \tag{B3}$$

one has

$$-\frac{\mathrm{d}}{\mathrm{d}t}\frac{1}{\beta L^d}E\ \ln Z_L(t) \geqslant \frac{\beta}{2}\sum_{r=2}^{p}\binom{p}{r}E\langle q_{12}^{p-r}y^r\rangle + \frac{3\beta}{8}\varepsilon_2^2 E\langle y^2\rangle \geqslant 0. \tag{B4}$$

As for the upper bound, the analogue of (41) can be conveniently rewritten as

$$\begin{aligned}
&-\frac{\mathrm{d}}{\mathrm{d}t}\frac{1}{\beta L^d}E\ \ln Z_L(t)\\
&= \frac{\beta}{2}\left(\frac{\ell}{L}\right)^d E\left(\ell^{d(p-1)}\sum_{n_1,\dots,n_p} w^{(p)}(i_{n_1},\dots,i_{n_p};\gamma)\langle q_{12}^{(n_1)}\cdots q_{12}^{(n_p)}\rangle - \sum_n\langle (q_{12}^{(n)})^p\rangle\right)\\
&\quad + \frac{\beta}{2}\sum_{s\geqslant 2}\varepsilon_s^2\left(\frac{\ell}{L}\right)^d E\left(\ell^{d(s-1)}\sum_{n_1,\dots,n_s} w^{(s)}(i_{n_1},\dots,i_{n_s};\gamma)\langle q_{12}^{(n_1)}\cdots q_{12}^{(n_s)}\rangle - \sum_n\langle (q_{12}^{(n)})^s\rangle\right)\\
&\quad + o(1)
\end{aligned}$$

where i_n is the lattice site situated at the centre of the nth box and the error term $o(1)$ vanishes in the Kac limit $\lim_{\gamma\to 0}\lim_{L\to\infty}$. Given a set of numbers $\tau^{(n)}$, where the index n runs over the cells Ω_n, define the average $\mu(\tau^{(\cdot)})$ as

$$\mu(\tau^{(\cdot)}) = \frac{\sum_n \psi(\gamma|i_n|)\tau^{(n)}}{\sum_n \psi(\gamma|i_n|)}. \tag{B5}$$

Then, it is not difficult to realize that

$$\begin{aligned}
-\frac{\mathrm{d}}{\mathrm{d}t}\frac{1}{\beta L^d}E\ \ln Z_L(t) = -\frac{\beta}{2}E\Big\langle &\left[\mu\left((q_{12}^{(\cdot)})^p\right) - \left(\mu(q_{12}^{(\cdot)})\right)^p\right]\\
&+ \sum_{s\geqslant 2}\varepsilon_s^2\left[\mu\left((q_{12}^{(\cdot)})^s\right) - \left(\mu(q_{12}^{(\cdot)})\right)^s\right]\Big\rangle + o(1).
\end{aligned} \tag{B6}$$

Now, given a measure μ and a bounded random variable $|q| \leqslant 1$, one has for $s \geqslant 3$,

$$|\mu(q^s) - (\mu(q))^s| \leqslant \xi_s(\mu(q^2) - (\mu(q))^2) \tag{B7}$$

for some constant ξ_s independent of μ and q. Then, it can immediately be seen that, if

$$\sum_{s=3}^{\infty} \varepsilon_s^2 \xi_s < \varepsilon_2^2 \tag{B8}$$

the average in (B6) is non-positive. The region R_p mentioned in section 2.2 is therefore defined by conditions (B3) and (B8).

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